Mathesis **III 3₂** (2008) 361 - 367.

The Pythagorean Theorem

Victor J. Katz

Eli Maor. 2007. *The Pythagorean Theorem: A 4,000-year History.* New Jersey: Princeton University Press. ISBN-13:978-0-691-12526-8

The Pythagorean Theorem is arguably the most important elementary theorem in mathematics, at least being recognized by anyone who has studied high school mathematics. In his book by the same name, Eli Maor has presented a well-written survey of 4000 years of the history of this theorem and related mathematical ideas. In his wonderful prose, he traces the theorem from Mesopotamia in 1800 BCE through Pythagoras, Euclid, and Archimedes in ancient Greece, François Viète around 1600 in France, and on to its relation to the space-time equations in relativity theory in the early 20th century. In general, the story is a fascinating one, showing the relationship of this theorem to many areas of mathematics. It will appeal to high school teachers as well as their students and certainly can be read with pleasure by anyone at all interested in mathematics.

Unfortunately, however, there are many problems with this book. Maor has failed to live up to the standard he set in several earlier works in the history of mathematics. Not only did he not use the latest scholarship, but he has in many cases misinterpreted the mathematical and historical material, thereby misleading his readers.

The first clue to the problems that this book presents is in the preface, where Maor notes that the main historical sources he uses are the sixth edition (1992) of Howard Eves' *An Introduction to the History of Mathematics* and David Eugene Smith's *History of Mathematics*. Not that there is anything particularly wrong with either of these books, but Smith's work was published in 1923-1925 while the 1992 edition of Eves' work is not all that different from his original work of 1953. Thus, although there have been numerous discoveries in the history of mathematics in the last 50-80 years impacting on the history of the

Pythagorean Theorem, Maor's sources, and therefore Maor's book, generally do not reflect these.

The Pythagorean Theorem probably makes its first appearance in the work of the Mesopotamian scribes close to four thousand years ago. There are several extant mathematical tablets from that time that show evidence of an understanding of this theorem, although without stating it explicitly or proving it. Maor describes the tablet YBC 7289 with it square of side 30 and its diagonal of length 42; 25,35 (*i.e.*, 42 + 25/60 + 35/3600) along with the approximation to the square root of two given as 1; 24,51,10, equivalent in decimals to 1.414212963…. Presumably, the scribe was aware that the diagonal of a square had a length equal to that value multiplied by the length of the side, a special case of the Pythagorean Theorem. But the theorem also occurs more generally in other Babylonian tablets, such as TMS 1, where the problem was to calculate the radius of a circle circumscribed about an isosceles triangle with altitude 40 and base 60. In this case, the scribe assumed the Pythagorean Theorem for the right triangle whose hypotenuse is the radius and whose legs are half the base of the original triangle and the line segment that is the difference between the given altitude and the radius. After some manipulation, the Pythagorean Theorem equation becomes a linear equation that is easily solved. Maor does not discuss this tablet, but instead concentrates on the more famous tablet Plimpton 322, which most scholars believe includes on each of its fifteen lines two numbers b and c that are parts of Pythagorean triples, triples of numbers (a,b,c) in which $c^2 = a^2 + b^2$. There has been much written about this tablet in the past sixty years since Otto Neugebauer first brought it to the world's attention. But Maor only reports on Neugebauer's own ideas, ideas that have been largely superseded by recent research. Thus, Maor claims that one possibility is that the tablet represents 'history's first trigonometric table', a table listing the squares of the cosecants of a sequence of fifteen angles ranging from 45 degrees to about 58 degrees. Secondly, he claims that that there is 'only one plausible explanation' as to how the ancient scribes figured out the numbers, by applying the algorithm later discussed in Euclid's *Elements*: $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$, for appropriately chosen integers u and v.

Yet modern researchers in Mesopotamian mathematics, including Eleanor Robson and Joran Friberg, have concluded, first of all, that there is no evidence of the concept of angle in that mathematics, hence no possibility of even considering a 'trigonometric' table. And second, they note, by actually translating the heading of one of the columns (that Maor characterizes as 'not entirely clear') and considering the general context of Mesopotamian mathematics, that the scribe calculated his values by considering reciprocal pairs. In other words, if we rewrite the basic Pythagorean relationship by subtracting b^2 from both sides and then dividing by a^2, we get the equation $w^2 - z^2 = 1$ (where $w = c/a$ and $z = b/a$), which can be factored as $(w + z)(w - z) = 1$, that is, as a pair of reciprocals. If one then reverses the procedure and starts with pairs of reciprocals, one can recover the Pythagorean triples [for more details, see Robson [2001] and Friberg [1981]].

Maor moves on to a discussion of Pythagoras and the Pythagoreans, dealing competently with their number theory and their presumed discovery of the incommensurability of the diagonal of a square with its side, related of course to the special case of the Pythagorean Theorem for isosceles right triangles. He then notes that we do not know what proof Pythagoras gave of the Pythagorean Theorem, speculating that it may have been similar to the one given in China a few hundred years later. (Of course, it is only tradition that allows us to believe that Pythagoras gave any proof of the theorem at all.) Maor then looks at Euclid's own proof (from *Elements* I-47) and asks why Euclid used this rather complicated argument rather than one of the simpler arguments presumably available to him. One of the standard answers is that many of these simpler arguments use proportion theory, and Euclid chose not to introduce that until later in his work. But as to why Euclid could not have used a cut and paste argument similar to that of the Chinese, Maor gives a curious answer, namely that a proof based on moving a plane figure would be 'anathema' to Euclid. In fact, however, Euclid did use proofs involving such motion to demonstrate two of the triangle congruence theorems in Book I of the *Elements*, although later commentators criticized Euclid for so doing. Secondly, given that the Chinese proof relies on comparing the areas of certain plane figures, it is not all that different from various proofs Euclid gives in Book II.

However Euclid decided on his proof, that the Pythagorean Theorem was proved to Greek standards now meant that it was part of the toolbox that Greek (and later Islamic) mathematicians could use in proving further theorems. Thus, Maor discusses how Archimedes used the theorem particularly in his *Measurement of the Circle*, in which the greatest mathematician of ancient times found the now-standard approximation 22/7 to the value of π by calculating the perimeters of various polygons inscribed in and circumscribed around a circle. Maor here, however, gives a misleading impression of Archimedes' work in that he describes his various approximations using decimals. Decimals, of course, were not used by the Greeks. And one of the fascinating parts of Archimedes' calculation is that he approximated various square roots by fractions. Although we are not certain of the actual algorithm, it was probably related to algorithms used in modern times to calculate square roots and thus would have made an interesting discussion by itself. But Maor also misses a wonderful chance to enlighten students about Archimedes' work, the *Method*. He briefly alludes in a footnote to the rediscovery of the palimpsest in which this work is found and its sale in 1998. But what happened after that sale is such an exciting story that it certainly bears inclusion in any popular book dealing with Archimedes at all [see Netz & Noel [2007] for details].

Chapter 5 of the book is entitled "Translators and Commentators, 500 – 1500," and deals with mathematics at the end of Greek period and then with Chinese, Indian, and Islamic mathematics. Although the three latter civilizations produced some excellent mathematicians, and even gave proofs and made use of the Pythagorean Theorem, Maor seems to adhere to the old story of the history of mathematics, that essentially nothing happened of interest in the time period he mentions besides translation and commentaries. He even repeats the discredited notion that Islamic mathematicians 'studied, interpreted, and evaluated' what they could find of ancient Greek mathematics and nothing else. In fact, as many recent histories of mathematics report, Islamic mathematicians carried on quite original research and achieved much in such fields as algebra, geometry, combinatorics, and trigonometry well before Europeans. As part of Maor's outdated historiography, he notes that Mohammed al-Khwarizmi was 'perhaps the greatest of all Arab mathematicians'. Now al-Khwarizmi is known for writing the first algebra text and for putting together an arithmetic work that ultimately introduced Europe to the Hindu decimal-place-value system. But he was not particularly an original mathematician. There were many 'greater' mathematicians who wrote in Arabic. A short list would include Abū 'Alī al-Hasan ibn al-Haytham (965-1039), Muhammad al-Bīrūnī (973-1055), Omar Khayyam (1048-1131), and Nasīr al-Dīn al-Tūsī (1201-1274). Curiously, even though Maor praises the Islamic mathematicians as translators of Greek mathematics, at the beginning of the chapter he notes that only six out of the thirteen books of Diophantus's *Arithmetica* are extant, totally ignoring the four additional books that were only preserved in Arabic translations. He also repeats the misleading story that the Arab conquerors of Alexandria in 641 burned the remnants of the famous library there. In all probability, there was nothing left of the library by that time, it having succumbed to various wars and invasions in the previous six centuries.

It is clear that Maor in writing chapter 5 did not avail himself of the plethora of new literature on the mathematics of China and India. If he had, he would have expanded this chapter considerably, because both civilizations made extensive use of the Pythagorean Theorem and even gave proofs of the result. For example, chapter 9 of the famous Chinese work *Nine Chapters on the Mathematical Art*, written around the beginning of our era and edited in the third century by Liu Hui, deals entirely with that result and its applications. And Liu Hui even supplies a proof somewhat different from the one Maor displays from the earlier *Arithmetical Classic of the Gnomon*. Both proofs depend on rearranging parts of plane figures and do not, as Maor notes, follow from explicit axioms. The Chinese did not follow Aristotle's prescriptions for logical argument. Nevertheless, both proofs can easily be turned into ones meeting Euclidean – or modern – standards. For more details on chapter 9 of the *Nine Chapters*, [see Swetz and Kao 1977]. Maor himself, however, only deals with one problem from that chapter, the problem of the broken bamboo, a problem that later appears in Indian and Renaissance European mathematics as well.

Maor's treatment of Indian mathematics is also quite superficial. In his two page treatment, Maor only mentions the early appearance of the Pythagorean Theorem in the *Sulvasutras* and a few of the construction problems based on it. He also comments that there is no information as to the proof that they gave. Certainly, the *Sulvasutras* themselves do not have proofs – that was not their purpose – but in the sixteenth century Jyesthadeva (1530-1610) did give a proof of the theorem, as well as of numerous other results. His proof, like the Chinese one, depended on manipulations with geometric figures and was not a Euclidean proof. Proof notwithstanding, the Indians did make much use of the theorem. It and its applications occur in many of the standard Indian texts, including works by Aryabhata (fifth century), Brahmagupta (seventh century) and Bhaskara (1114-1185). For example, Bhaskara devoted several verses to the theorem in his *Lilavati*. In particular, he showed how to find the three sides of a right triangle if one knew one leg and the sum of the hypotenuse and the other leg. He then applied this result to solve the following: One monkey came down from a tree of height one hundred and went to a pond at a distance of two hundred. Another monkey, leaping some distance above the tree, went diagonally to the same place. If their total distances traveled are equal, how much is the height of the leap? [For more examples, see Plofker [2007]].

It is unfortunately indicative of Maor's reluctance to consult new sources that he writes in a footnote to chapter 5 that he has taken his material from George Joseph's *Crest of the Peacock*, noting further that the subject is 'only marginally covered in most books on the history of mathematics'. In fact, several recent texts in the history of mathematics devote substantial space to Indian mathematics [see Katz [1998], [2004], [2009], Cooke [1997], [2005], Suzuki [2002]]. And even a thorough reading of George Joseph's book would have prevented Maor from his error in a footnote on p. 96, where he fails to credit the Indian mathematician Madhava for the famous series $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ also discovered by James Gregory and Wilhelm Leibniz.

Once Maor moves on to European mathematics, his historiography seems to be freer of error, and he has wonderful chapters on Viète's computation of an infinite product representation of $2/\pi$, on the use of the Pythagorean Theorem to calculate curve lengths in calculus in the seventeenth century, and on the many proofs that have been given of the theorem, many of which were collected by Elisha Scott Loomis in his *The Pythagorean Proposition* (1927). But even in some of these later chapters there are errors. For example, in his discussion of the Law of Cosines, Maor incorrectly quotes its first appearance in Euclid's *Elements* and later confuses Riemann's doctoral dissertation with his more famous habilitation lecture. But there are also interesting discussions of the principle of duality in projective geometry, various notations that can be used to state the Pythagorean Theorem and its generalizations, and the use of variations of the theorem in the theory of relativity. Interspersed among the chapters of the book are ten sidebars, short essays dealing with some interesting point, such as a trigonometric proof of the theorem or the appearance of the theorem in art and literature.

Despite the many historiographical errors, Maor's book is still worth reading. It has the potential of exciting students and getting them to understand that mathematical theorems are not just dry statements given from on high. Theorems such as the Pythagorean Theorem have long histories, were developed by real people, and have had many applications in all sorts of areas. But if a teacher were to suggest that students read this book, it is imperative that the teacher be aware of its imperfections so that he or she can help the students find the errors and correct them. I also hope that Maor himself would take these criticisms seriously and prepare a second revised edition taking these into account.

References

COOKE, Roger. 1997, 2005. *The History of Mathematics: A Brief Course*, 1st & 2nd editions. New York: John Wiley & Sons.

FRIBERG, Joran. 1981. "Plimpton 322, Pythagorean triples, and the Babylonian triangle parameter equations". *Historia Mathematica* **8**: 277-318.

KATZ, Victor J. 1998, 2009. *A History of Mathematics: An Introduction*, 2nd & 3rd editions. Boston: Addison-Wesley.

———. 2004. *The History of Mathematics: Brief Version*. Boston: Addison-Wesley.

NETZ, Reviel and William NOEL. 2007. *The Archimedes Codex: Revealing the Secrets of the World's Greatest Palimpsest*. London: Weidenfeld and Nicolson.

PLOFKER, Kim. 2007. "Mathematics in India", in Victor J. Katz, ed., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton: Princeton University Press, pp. 385-514.

ROBSON, Eleanor. 2001. "Neither Sherlock Holmes nor Babylon: A reassessment of Plimpton 322". *Historia Mathematica* **28**: 167-206.

SUZUKI, Jeff. 2002. *A History of Mathematics*. Upper Saddle River, NJ: Prentice Hall.

SWETZ, Frank and T. I. KAO. 1977. *Was Pythagoras Chinese?* University Park, Pa.: Pennsylvania State University Press.